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CORRECTIONS TO "ON SEQUENTIAL CONVERGENCE"

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In my paper [2], Theorems 8.2 and 8.3 are false. I am indebted to Dennis Sentilles who sent me a counterexample to Theorem 8.2.

The wrong statements concern "*LS*-spaces." These are linear spaces S on which there is a metric ρ and a function f such that $x_n \to x$ in S if and only if both $\rho(x_n, x) \to 0$ and $f(x_n)$ remains bounded. Without restating the further conditions imposed on ρ and f, we single out for the present two tractable subclasses of the class of *LS*-spaces:

(I) *LF*-spaces, i.e. strict inductive limits of Fréchet spaces ([1], [2, Theorem 7.7]).

(II) If B is a separable Banach space, and S is its dual B^* with weak-star convergence of sequences, then S is an *LS*-space [2, Theorem 7.1]. (Actually the same is true with "Fréchet" in place of "Banach": see [1, Théorème 5, p. 84].)

In case (I), Theorems 8.2 and 8.3 are true. The Banach-Steinhaus theorem for *LF*-spaces is well known [1, Théorème 2, p. 73]. But in case (II), if B is infinite-dimensional, the elements of its unit ball give pointwise bounded continuous linear forms on S which are not equicontinuous for the *LS*-topology. The proof of Theorem 8.2 in [2] errs in assuming that multiplication by a positive scalar is an open mapping in the relative topology of a (convex, symmetric, closed, metrizable) set.

The proof of Theorem 8.3 is invalid since it rests on Theorem 8.2. Disproving the statement of 8.3 requires some further work. Here is one counterexample.

Let $\mathscr{C}$ be the space of continuous real functions on [0, 1] with supremum norm. Then its dual $\mathscr{C}^*$ is the space of finite signed measures on [0, 1]. $\mathscr{C}^*$ with weak* sequential convergence is an *LS*-space. It has a countable dense set and is complete as an *LS*-space. The *LS*-topology $\mathscr{T}$ on $\mathscr{C}^*$ is the topology of uniform convergence on sequences $\{f_n\}$ in $\mathscr{C}$ with $\|f_n\| \to 0$ [2, around Theorem 7.8]. By the Mackey-Arens theorem, the dual space of $(\mathscr{C}^*, \mathscr{T})$ is $\mathscr{C}$ (see also [1, Théorème 6, p. 85]).

A sequence in $\mathscr{C}$ is weakly convergent if and only if it is uniformly bounded and converges pointwise. Thus the following fact contradicts Theorem 8.3:

PROPOSITION. *Bounded pointwise convergence in $\mathscr{C}$ is not countably quasi-metric.*

Proof. Suppose there is a metric d and a countable set $\{G_m\}_{m=1}^{\infty}$ of functions on $\mathscr{C}$ such that if $\|f_n\| \leq 1$, then $f_n \to 0$ pointwise if and only if both:

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(a) $d(f_n, 0) \to 0$ and

(b) for each m, $\sup_n G_m(f_n) < \infty$.

For some intervals $I_m = [a_m, b_m]$, $0 = a_0 < a_1 < \cdots < a_m < \cdots < b_m < \cdots < b_1 < b_0 = 1$, let $\mathscr{C}_m = \{f \in \mathscr{C} : \|f\| \leqq 1, f(x) = 0 \text{ for all } x \notin I_m\}$. We can define a_m and b_m inductively so that for each m,

(c) $\sup\{G_m(f) : f \in \mathscr{C}_m\} < \infty$ and

(d) $\sup\{d(0,f) : f \in \mathscr{C}_m\} < 1/m$.

To do this, we can let $a_{m+1} = (a_m + b_m)/2$ and $b_{m+1} = a_{m+1} + 1/r$ for r large enough (if $g_r \in \mathscr{C}$, $\|g_r\| \leqq 1$, and $g_r = 0$ outside $[a_{m+1}, a_{m+1} + 1/r]$, then $g_r \to 0$ pointwise).

Now there is a c with $a_m < c < b_m$ for all m, and there are f_m in $\mathscr{C}_m$ with $f_m(c) = 1$ for all m. But (a) through (d) imply $f_m(c) \to 0$, a contradiction. Q.E.D.

Another correction: on p. 506 in [2], it is wrongly stated that topologies $T(C)$ on distribution spaces $\mathscr{E}'$ and $\mathscr{S}'$ are not locally convex (see [3]).

References

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3. J. H. Webb, *Sequential convergence in locally convex spaces*, Proc. Cambridge Philos. Soc. **64** (1968), 341–364. MR **36** #5652.

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